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POLYTOPAL AND NONPOLYTOPAL SPHERES AN ALGORITHMIC APPROACH

BY

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ABSTRACT

The convexity theory for oriented matroids, first developed by Las Vergnas [17], provides the framework for a new computational approach to the Steinitz problem [13]. We describe an algorithm which, for a given combinatorial $(d-2)$-sphere S with n vertices, determines the set $C_{d,n}(S)$ of rank d oriented matroids with n points and face lattice S. Since S is polytopal if and only if there is a realizable $M \in C_{d,n}(S)$, this method together with the coordinatizability test for oriented matroids in [10] yields a decision procedure for the polytopality of a large class of spheres. As main new result we prove that there exist 431 combinatorial types of neighborly 5-polytopes with 10 vertices by establishing coordinates for 98 "doubted polytopes" in the classification of Altshuler [1]. We show that for all $n \geqq k+5 \geqq 8$ there exist simplicial k-spheres with n vertices which are non-polytopal due to the simple fact that they fail to be matroid spheres. On the other hand, we show that the 3-sphere M^9_{963} with 9 vertices in [2] is the smallest non-polytopal matroid sphere, and non-polytopal matroid k-spheres with n vertices exist for all $n \geqq k+6 \geqq 9$.

1. Introduction

The classification of all combinatorial types of convex polytopes of given dimension and number of vertices has a long tradition in combinatorial convex geometry. While this problem is completely solved for 3-polytopes by Steinitz' theorem [15] and for d-polytopes with less than $d+4$ vertices by results of Mani and Kleinschmidt, see [13], still only very little is known for d-polytopes with n vertices where $n \geqq d+4 \geqq 8$.

All attempts to solve the Steinitz problem, i.e., to find intrinsic characterizations for boundary complexes of higher-dimensional polytopes, have not been successful. Considering the results of this paper it seems very likely that such a local Steinitz theorem does not exist.

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In 1967 Grünbaum [15] gave an affirmative answer to the question whether there exists at least an algorithm that enumerates all combinatorial types of spheres with n vertices. Enumerating all combinatorial spheres one is left with the problem to decide whether a given sphere is isomorphic to the boundary complex of a polytope in $\mathbf{R}^d$. This problem can be formulated in terms of elementary algebra. Therefore it is decidable by a theorem of Tarski [23]. Tarski's algorithm, however, is hopelessly inefficient and so far no efficient decision procedure for the polytopality of combinatorial spheres is known.

This paper deals with the transformation of this problem to the realizability problem of oriented matroids.

Throughout the paper an oriented matroid will be represented by its chirotope, i.e., the set of signed bases, compare [9].

In [10] we describe an efficient method to solve inequality systems arising from chirotopes. This algorithm finds coordinates for a large class of chirotopes and so far no realizable configuration not belonging to this class is known.

Thus this paper should be seen in connection with [9] and [10], although those papers are written in terms of the more general theory of oriented matroids. The development of our method was mainly motivated by the Steinitz problem and has its origin in proofs of polytopality and non-polytopality due to the first author in [6], [8], [7].

After giving the basic concepts of convexity in oriented matroids in terms of their chirotopes in Section 2, we describe in Section 3 how to find the set of d-chirotopes with a face-lattice isomorphic to a given sphere. In Section 4 we prove the existence of a wide class of non-matroid spheres and in Section 5 we complete the classification of all neighborly 4-polytopes with 10 vertices. Finally, in Section 6 we treat the case of non-polytopal simplicial matroid spheres.

2. Oriented matroid spheres

In this section we recall the basic concepts of convexity in oriented matroids in terms of their chirotopes. For details see [5], [17] and the other papers quoted in [9].

Let $\Lambda(n,d):=\{(\lambda_1,\ldots,\lambda_d)\in\mathbf{N}^d \mid 1\leqq\lambda_1<\cdots<\lambda_d\leqq n\}$ be the set of ordered d-tuples of n elements.

$\lambda\in\Lambda(n.d)$ will be considered sometimes as set $\lambda=\{\lambda_1,\ldots,\lambda_d\}$ as well. A mapping $\chi\colon\Lambda(n,d)\to\{-1,0,+1\}$ is called a *d-chirotope with n vertices* if for all $\mu\in\Lambda(n,d+2)$ there exist vectors $x_{\mu_1},\ldots,x_{\mu_{d+2}}\in\mathbf{R}^d$ such that for all $\lambda\in\Lambda(n,d)$ with $\lambda\subset\mu$

$$\chi(\lambda) = \text{sign}(\det(x_{\lambda_1}, \ldots, x_{\lambda_d})).$$

χ will be extended to $\{1, \ldots, n\}^d$ in the canonical alternating way.

It is easy to see that this definition is equivalent to those given in [9] using hyperline images or Grassmann–Plücker relations.

A d-chirotope χ with n vertices is called *realizable* if there exist $x_1, \ldots, x_n \in \mathbf{R}^d$ such that for all $\lambda \in \Lambda(n, d)$

$$\chi(\lambda) = \text{sign}(\det(x_{\lambda_1}, \ldots, x_{\lambda_d})).$$

In this case χ is also called the *chirotope of linear dependencies* on $\{x_1, \ldots, x_n\}$.

Thus the definition of a d-chirotope says each choice of a restriction of χ corresponding to $\mu \in \Lambda(n, d+2)$ is realizable.

The $(k+1)$-*chirotope of affine dependencies* on $\{y_1, \ldots, y_n\} \in \mathbf{R}^k$ is the chirotope of linear dependencies on $\{1\} \times \{y_1, \ldots, y_n\} \in \mathbf{R}^{k+1}$.

χ is called *simplicial* if $\operatorname{Im} \chi \subset \{-1, +1\}$.

The *cocircuits of* χ are the signed vectors

$$\pm C_\mu(\chi) := \pm(\ldots, \chi(\mu_1, \ldots, \mu_{d-1}, i), \ldots) \in \{-1, 0, +1\}^n \quad \text{for } \mu \in \Lambda(n, d-1).$$

Given a d-chirotope χ with n vertices the *dual of* χ is defined by

$$\chi^*: \Lambda(n, n-d) \rightarrow \{-1, 0, +1\}$$
$$\lambda \mapsto \chi(\{1, \ldots, n\} \setminus \lambda) \cdot (-1)^{\sum_{i=1}^{n-d} \lambda_i}$$

The *circuits of* χ are cocircuits of χ^*.

We define the *deletion* $\chi \setminus n := \chi\big|_{\Lambda(n-1,d)}$ and the *contraction* $\chi / n := (\chi^* \setminus n)^*$ for the vertex n. The general case we get by renumbering.

$F \subset \{1, \ldots, n\}$ is called a *facet* of χ if there exists a cocircuit $C_\mu(\chi) \in \{+1, 0\}^n$, $\mu \in \Lambda(n, d-1)$, with

$$\chi(\mu_1, \ldots, \mu_{d-1}, i) = \begin{cases} 0 & \text{for } i \in F \\ 1 & \text{for } i \in \{1, \ldots, n\} \setminus F \end{cases}$$

or in short notation

$$C\big|_F \equiv 0 \quad \text{and} \quad C\big|_{\{1, \ldots, n\} \setminus F} \equiv 1.$$

$F \subset \{1, \ldots, n\}$ is called a *face* of χ if $F = \bigcap_{i=1}^k F_i$ for facets F_i, $1 \leqq i \leqq k$, of χ.

The face lattice $\mathrm{FL}(\chi)$ is the lattice of all faces ordered by inclusions. χ is called *affine* if $\phi \in \mathrm{FL}(\chi) \Leftrightarrow \mathrm{FL}(\chi^*) = \emptyset$.

Given a convex k-polytope $P = \operatorname{conv}\{y_1, \ldots, y_n\} \subset \mathbf{R}^k$ the face lattice of P is isomorphic to the face lattice $\mathrm{FL}(\chi)$ of the chirotope χ of affine dependencies on $\{y_1,\ldots,y_n\}$. Any realization of χ^* is isomorphic to a Gale transform of P.

For any linear realization $\{x_1,\ldots,x_n\}\subset\mathbf{R}^{k+1}$ of χ the polyhedral cones $\mathrm{pos}\{x_1,\ldots,x_n\}$ and $\mathrm{pos}(P\times\{1\})\subset\mathbf{R}^{k+1}$ have the same face lattice $\mathrm{FL}(\chi)$.

A combinatorial sphere S is called a *matroid sphere* if $S=\mathrm{FL}(\chi)$ for a chirotope χ. Clearly, S is polytopal if and only if $S=\mathrm{FL}(\chi)$ for a realizable chirotope χ.

We shall see later in this paper that there exist non-polytopal matroid spheres as well as combinatorial spheres which already fail to be an oriented matroid sphere.

Since every matroid sphere is a combinatorial PL-sphere by a result of Mandel [18] it is a consequence of Steinitz' theorem that $\mathrm{FL}(\chi)$ is polytopal for every d-chirotope with $d\leqq 4$.

The Vamòs matroid [5] is a smallest example of a non-realizable chirotope which has only extreme points such that its face lattice is polytopal. Thus in order to prove the non-polytopality of a sphere S one has to prove the non-realizability of all chirotopes χ with $S=\mathrm{FL}(\chi)$.

3. Chirotopes arising from a given sphere

If we had a subroutine to decide the realizability of a chirotope, the following algorithm would decide the polytopality of a given sphere.

3.1. ALGORITHM.

INPUT: Combinatorial $(d-2)$-sphere S with n vertices

OUTPUT: "S is polytopal" or "S is not polytopal"

1. Compute the finite set

$$C_{d,n}(S):=\{\chi\mid\chi\ d\text{-chirotope with } n \text{ vertices and } \mathrm{FL}(\chi)=S\}$$

2. If there is a realizable $\chi\in C_{d,n}(S)$: "S is polytopal"

Else "S is not polytopal"

In this section we shall describe the first step, the computation of the set $C_{d,n}(S)$. For an approach towards the hard second step, see [10].

First let us assume S to be simplicial. Then we can identify S with the subset $F_s\subset\Lambda(n,d-1)$ of facets of S.

Let $S_s:=\{\mu\in\Lambda(n,d)\mid\exists\mu_i:\ \mu\setminus\mu_i\in F_s\}$ denote the set of outer simplices according to S. To verify that $\chi|_{S_s}$ is determined by $S=\mathrm{FL}(\chi)$ consider the graph $G_S = (S_s, E)$ on the outer simplices with

$$(\sigma, \tau) \in E :\Leftrightarrow \sigma \cap \tau \in F_s.$$

G_S is connected: For fixed $\mu \in F_s$, $G_S\,|_{\{(\mu_1,\dots,\mu_{d-1},i) | i \notin \mu\}}$ is obviously connected. On the other hand, the components $G_S\,|_{\{(\mu i) | i \notin \mu\}}$ are connected to each other by a graph which is isomorphic to the 1-skeleton of the dual sphere S^*.

As a direct consequence of the definitions of cocircuits and the face lattice of χ we have

$$(*) \qquad \chi \in C_{d,n}(S) \Leftrightarrow \text{for all } \mu \in F_s \text{ and } i, j \notin \mu : \chi(\mu_i)\chi(\mu_j) = 1.$$

Now choosing a spanning tree in G_S and defining w.l.o.g. $\chi(\sigma) := +1$ for the root $\sigma \in S_s$ any one of the usual tree-traversing algorithms will assign the vertices τ of G_S the correct sign $\chi(\tau)$ according to (*). Thus $\chi\,|_{S_s}$ is determined by S.

Now we have to extend $\chi\,|_{S_s}$ to a chirotope (if it exists). This can be done by using consequences of the quadratic Grassmann–Plücker relations. The demand that χ should be a chirotope is equivalent to the property that for any $\sigma \in \Lambda(n, d-2)$, $\tau \in \Lambda(n, 4)$ the set

$$\{\chi(\sigma\tau_1\tau_2) \cdot \chi(\sigma\tau_3\tau_4), \chi(\sigma\tau_1\tau_3) \cdot \chi(\sigma\tau_2\tau_4), \chi(\sigma\tau_1\tau_4) \cdot \chi(\sigma\tau_2\tau_3)\}$$

equals $\{0\}$, $\{-1, +1\}$ or $\{-1, 0, +1\}$, compare [9]. The following example will illustrate this.

3.2. Example. Let S be a hexagon, i.e., $F_s = \{(12), (23), (34), (45), (56), (16)\} \subset \Lambda(6, 2)$. By (*), $\chi(123) := +1$ implies $\chi\,|_{S_s} \equiv +1$, whereby $S_s = \Lambda(6,3) \setminus \{(135), (246)\}$. With $\sigma = (1) \in \Lambda(6, 1)$, $\tau = (2345) \in \Lambda(6, 4)$ we have

$$\{\chi(123) \cdot \chi(145), -\chi(124) \cdot \chi(135), \chi(125) \cdot \chi(134)\} = \{+1, -\chi(135)\}.$$

This yields $\chi(135) = +1$. Similarly $\chi(246) = +1$ for every chirotope χ with $\mathrm{FL}(\chi) = S$.

While for n-gons S there is, apart from sign reversing, only one 3-chirotope χ with n vertices such that $S = \mathrm{FL}(\chi)$, in general there will be several chirotopes in $C_{d,n}(S)$.

In this case the Grassmann–Plücker consequences will only lead to $\chi\,|_A$ with $S_s \subset A \subsetneq \Lambda(n, d)$ and it is necessary to traverse a tree of possible extensions of $\chi\,|_A$ to a chirotope.

Finally we consider the case of a non-simplicial sphere. In this case we compute a triangulation S' of S. This can be done, recursively, by determining

$(d-1)$-chirotopes χ_F with $\mathrm{FL}(\chi_F)=F$ for every facet F of S. Any simplicial lift χ'_F of χ_F, i.e., a simplicial d-chirotope with $\chi'_F/(n+1)=\chi_F$, leads to an oriented matroid triangulation of F, compare [4].

Now we compute $\chi\,|_{S'}$ for the simplicial sphere S' as described above and redefine $\chi(\lambda)=0$ if $\lambda \subset F$ for a non-simplicial facet of S. Again extending $\chi\,|_{S'}$ to possible chirotopes χ we obtain the desired set $C_{d,n}(S)$.

4. Non-matroid spheres

For $k \leqq 2$ or $n \leqq k+4$ every combinatorial k-sphere with n vertices is polytopal, hence an oriented matroid sphere, compare Mani [19] and Kleinschmidt [16]. In this section we prove the reverse.

4.1. Theorem. *There exist non-matroid k-spheres with n vertices whenever $k \geqq 3$ and $n \geqq k+5$.*

Proof. Consider the Barnette sphere $\mathscr{B}$, i.e., the 3-sphere with 8 vertices given by the following list of facets:

$\underline{1237}$	1258	1458	2368	3567
1238	$\underline{1346}$	$\underline{1467}$	$\underline{2457}$	4567
$\underline{1245}$	1348	$\underline{2356}$	2568	4568
$\underline{1247}$	1367	$\underline{2357}$	3468	

We prove that the subcomplex $\mathscr{C}$ determined by the eight underlined facets cannot be the subcomplex of any matroid sphere $\mathrm{FL}(\chi)$.

If there were a 5-chirotope χ such that $\mathscr{C} \subset \mathrm{FL}(\chi)$ then (*) would imply that in χ the following 12 simplices have the same non-zero orientations:

$$\begin{array}{ccc}
12367 & & 12543 \\
| & & | \\
12357 & & 12547 \\
| & & | \\
12347 & = & 12347 \\
| & & | \\
12547 & & 12647 \\
| & & | \\
32547 & & 13647 \\
| & & | \\
32567 & & 13642 \\
| & & \\
32561 & &
\end{array}$$

Choosing $\sigma = (123) \in \Lambda(8,3)$, $\tau = (4567) \in \Lambda(8,4)$ we have

$$\{\chi(\sigma\tau_1\tau_2)\chi(\sigma\tau_3\tau_4), -\chi(\sigma\tau_1\tau_3)\chi(\sigma\tau_2\tau_4), \chi(\sigma\tau_1\tau_4)\chi(\sigma\tau_2\tau_3)\}$$
$$= \{\chi(12345)\chi(12367), -\chi(12346)\chi(12357), \chi(12347)\chi(12356)\} = \{-1\}$$

contradicting the chirotope property.

By the usual techniques of subdividing facets and constructing pyramids, [15], we can extend $\mathcal{B}$ to a combinatorial non-matroid k-sphere with n vertices for all $n \geqq k+5 \geqq 8$. □

4.2. REMARK. A similar proof for both the non-polytopal Brückner sphere [15] and the Barnette-sphere can be found in [3]. The fact that every non-polytopal 3-sphere with 8 vertices is not a matroid sphere can be obtained by the following indirect argument, too.

Every 3-chirotope with 8 vertices is reorientation equivalent (compare [17]) to an acyclic 3-chirotope with 8 vertices and therefore corresponds to a stretchable arrangement of 8 pseudolines [14], [11]. By dualizing we see that every 5-chirotope with 8 vertices is realizable, hence every matroid 3-sphere with 8 vertices is polytopal.

5. Neighborly 4-polytopes with 10 vertices

In [1] Altshuler established an enumeration of all combinatorial neighborly 3-spheres with 10 vertices and classified those 3-spheres as far as possible into polytopes and non-polytopal spheres. He proved that there exist at least 333 and at most 432 neighborly 4-polytopes with 10 vertices.

Using the methods described in the preceding sections and the coordinization algorithm described in [10] we elaborated on the 99 doubted polytopes and achieved the following result.

5.1. THEOREM. *There are precisely* 431 *different combinatorial types of neighborly* 4-*polytopes with* 10 *vertices.*

5.2. REMARKS. For the sphere M^{10}_{425} (Altshuler's notation) which is of particular interest because of its high symmetry, a proof for non-polytopality is given in [7].

This proof has been established by the same technique as the proof for a smallest non-polytopal matroid sphere in the next section.

Among the new decided spheres was another case of special interest, sphere

M^{10}_{416}. In this case the combinatorial automorphism was used in addition. It lead to a first neighborly 4-polytope known to have no *universal edge*. This answers a question of Perles, posed again in Oberwolfach 1984. Universal edges have been used by Shemer to obtain a huge family of neighborly polytopes with his *sewing process* [21].

In oriented matroid theory universal edges have been introduced under the name of contravariant pairs of points [12]. For a chirotope $\chi \in \{-1, 0, +1\}^{\Lambda(n,d)}$ the pair (i, j), $i, j \in \{1, \dots, n\}$, is called a *contravariant pair*, if i and j have the same sign in every cocircuit of χ, i.e., i and j are not separated by any spanned hyperplane $H \subset \{1, \dots, n\} \setminus \{i, j\}$.

Our procedure to decide all the remaining doubted polytopes yielded projectively unique chirotopes. The uniqueness in these cases was shown by Shemer in general, see [21], theorem 2.12. In our notation Shemer's results reads:

Let S be the boundary complex of a neighborly $2k$-polytope. Then $C_{2k+1,n}(S)$ contains exactly two simplicial chirotopes χ_S and $-\chi_S$.

A new proof of Shemer's result in the broader setting of oriented matroids has recently been found by the second author and will be published in a forthcoming paper.

5.3. *Description of Finding the Proof of our Theorem According to Algorithm 3.1*

For all 99 former doubted polytopes we found in the first step of algorithm 3.1 the corresponding projectively unique chirotopes. Thus the problem remained to solve 99 inequality systems with $\binom{10}{5} = 252$ determinant inequalities. Using the concept of solvability sequences as described in [10] the most interesting cases M^{10}_{425} and M^{10}_{416} (in Altshuler's notation) [1] were decided first. For M^{10}_{425} see [7]. A more detailed exposition of proofs for non-realizability is planned for a forthcoming paper.

The remaining 97 spheres have universal edges (i, j). This means algebraically that according to a basis $b \in \Lambda(10, 5)$ with $i \in b$, $j \notin b$, the variable $(b)\,[i \mid j]$ has either no upper bound or no lower bound. Hence at least $\binom{8}{4} = 70$ of the 252 inequalities can be omitted, immediately.

In the following we don't give again a detailed description of the solvability sequence algorithm [10] which was used, but we briefly introduce the main ideas with the following example of the cyclic 4-polytope with 10 vertices. For notations compare [10].

Example. Let $\chi\colon \Lambda(10,5) \to \{+1\}$ be the cyclic 5-chirotope with 10 vertices. For every 10×5 matrix M with $\Lambda_5 M = \chi$ there exists a 5×5 matrix T such that

$$M \cdot T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & b & c & d & e \\ f & g & h & i & j \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ k & l & m & n & o \\ p & q & r & s & t \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ u & v & w & x & y \end{pmatrix}.$$

The variable a in the standard representative matrix $M \cdot T$ according to the basis (14589) $\Lambda(10, 5)$ equals the positive determinant (14589)[1/2] = (24589) and has no upper bound in the induced inequality system with variables $\{a, \dots, y\}$ since $(1, 2)$ is a contravariant pair (universal edge) in χ.

We call $(y, t, o, j, e, x, s, n, i, d, w, r, m, h, c, v, q, l, g, b, u, p, k, f, a)$ a *solvability sequence* for χ according to the basis (14589) because in choosing real numbers for the variables, *every* choice of the first z numbers, $1 \leqq z \leqq 24$, which is allowable with respect to the inequality system still allows one to find a real number for the next variable.

In case of our doubted polytopes we used a computer program to find solvability sequences. 59 of all doubted polytopes were even max-realizable in the sense of [10] (Section 5). In 9 cases we finished our investigation after having found a "nearly" solvability sequence with the aid of an interactive computer program which lead to coordinates.

A detailed list of all 98 polytopes can be obtained upon request from the authors. Here we illustrate only one special example.

5.4. Example M^{10}_{360}. The facets of this neighborly sphere are

1234	1279	2340	3459	4567
1236	1346	2369	3467	4560
1240	1460	2390	3478	4578
1258	1580	2589	3490	4590
1250	1678	2590	3589	5670
1269	1679	2789	3679	5780
1278	1680	3458	3789	6780

The corresponding (pair of) chirotope(s) was found by a first program to be

$\chi^{10}_{360}(\Lambda(10,5))$ as follows:

$$\chi^{10}_{360}(12345) = -1, \quad \chi^{10}_{360}(12346) = -1, \quad \ldots, \quad \chi^{10}_{360}(67890) = +1.$$

The whole list ordered linewise lexicographically is:

```
----------  ++++++-+-+  +----+++++  +-+-+++++-  +-----+-+-
++++-----+  ++++++++++  ++++-++-+-  ++-+-++-++  ++++-++-+-
++-+-++-++  ++-+-+--+-  ---+++---+  ++++++++++  ++++-++-+-
++-+-++-++  ++++-++-+-  ++-+-++-++  ++-+-+--+-  ---+++++--
+----+++++  ++++++++++  +++++++---  -+++++++++  ++----+---
-+
```

A best basis with respect to the number of steps in which the concept of max-realizability works [10] was determined by a second program to be (12367). Thus the first 6 steps of the solvability sequence were found, namely (12467), (12368), (12365), (12369), (12360), (12364). To get the new inequality system exactly all determinants containing the above variables (compare the basis (12367)) were deleted.

This remaining inequality system was now reduced, that is to say, inequalities which were consequences of the remaining inequality system (by using Grassmann–Plücker relations), see the following list, were deleted.

(12780)	(12790)	(13579)	(13789)	(13780)	(13790)	(15679)
(15789)	(15780)	(15790)	(16789)	(16790)	(17890)	(23579)
(23789)	(23780)	(23790)	(25679)	(25789)	(25780)	(25790)
(26789)	(26790)	(27890)	(34570)	(34789)	(34780)	(35679)
(35789)	(35780)	(35790)	(36789)	(36790)	(37890)	(56789)
(56790)	(57890)	(67890)				

We were left with an inequality system (minimal system) of 14 inequalities together with 19 inequalities for the variables itself in 19 variables.

The remaining determinants are

(12570)	(12789)	(15670)	(16780)	(23479)	(23470)	(23578)
(34567)	(34578)	(34579)	(34679)	(34790)	(35670)	(56780)

The next 2 variables (12967), (12867) were eliminated according to algorithm 4.2 (3.1) in [10] whereby none of the disjoint subsets of $\Lambda(10,5)$ were empty, marked by (*) in the following list.

The whole solvability sequence is given in Table 1 as well as the list of homogeneous coordinates.

TABLE 1

Solvability sequence			Coordinates
25	(12467)	:=	31.00
24	(12368)	:=	2817.45
23	(12365)	:=	207.00
22	(12369)	:=	11.00
21	(12360)	:=	3.00
20	(12364)	:=	1.00
19	(12967)	:=	−0.01
18	(12867)	:=	0.59
17	(18367)	:=	−1.17
16	(12387)	:=	−29.00
15	(82367)	:=	1.00
14	(12357)	:=	−7.00
13	(12567)	:=	3.00
12	(12067)	:=	1.00
11	(52367)	:=	0.50
10	(92367)	:=	−2.00
9	(02367)	:=	0.67
8	(42367)	:=	1.00
7	(15367)	:=	−1.00
6	(19367)	:=	7.00
5	(14367)	:=	−3.00
4	(10367)	:=	−1.00
3	(12347)	:=	−1.00
2	(12397)	:=	1.00
1	(12307)	:=	−1.00

Any 4-hyperplane H separating $(0,0,0,0,0)$ from all vectors $(v_{i1},\dots,v_{i5})$, $i=1,\dots,10$ now defines a polytope

$$P=\operatorname{conv}\{H\cap L_1,\dots,H\cap L_n\},\quad L_i:=\{\lambda(v_{i1},\dots,v_{i5})\mid\lambda>0\}$$

with the given list of facets.

6. Non-polytopal matroid spheres

For a certain time after the introduction of face lattices for oriented matroids by Las Vergnas [17] and, in a polar version, by Edmonds and Mandel [18], it was not known whether there exist any non-polytopal matroid spheres. In 1980 Lawrence found a construction that yielded an affirmative answer to this question, see [20]. The smallest example that can be constructed with Lawrence's method is a 10-sphere with 16 vertices.

For simplicial spheres which can never be obtained by Lawrence's extension and for k-spheres with $3\leqq k\leqq 9$ the problem remained unsolved.

In this section we show that there are non-polytopal simplicial matroid k-spheres with n vertices for $3 \leqq k$ and $n \geqq k+6$ by proving that the sphere M^9_{963} from [2] is a smallest example of a non-polytopal matroid sphere.

The non-realizability proof for the chirotope of M^9_{963} might be of interest on its own because it indicates a general method to prove non-realizability of oriented matroids.

We begin with a brief discussion of Lawrence's method in the terminology of Sections 2 and 3.

Let $\chi: \Lambda(n,d) \to \{-1,0,+1\}$ be a chirotope. We define a new chirotope $\hat{\chi}: \Lambda(2n,d) \to \{-1,0,+1\}$ as follows. Write $\lambda \in \Lambda(2n,d)$ as $\lambda = \mu + r \cdot n$ where $\mu_i = \lambda_i \mod n$, $1 \leqq \mu_i \leqq n$. Then

$$\hat{\chi}(\lambda) := \chi(\mu) \cdot (-1)^{\sum_{i=1}^{d} r_i}.$$

In case of realizability, i.e. $\chi = \operatorname{sign} \Lambda_d M$, we have

$$\hat{\chi} = \operatorname{sign} \Lambda_d \begin{pmatrix} M \\ -M \end{pmatrix}.$$

$(\hat{\chi})^*$ (the dual of $\hat{\chi}$) is called the *Lawrence extension* of χ.

6.1. THEOREM (compare [20], Theorem 3.1.2). *χ is realizable if and only if $\mathrm{FL}(\hat{\chi}^*)$ is polytopal.*

PROOF. We assume that $S = \mathrm{FL}(\hat{\chi}_1^*) = \mathrm{FL}(\hat{\chi}_2^*)$ for two d-chirotopes χ_1 and χ_2 with n vertices. By construction $\{i, i+n\}$ is a cofacet of S for every $i \in \{1, \ldots, n\}$.

Since for every $\mu \in \Lambda(n, n-d)$ and $j \notin \mu$, $\{\mu_1, \ldots, \mu_{n-d}, n+1, \ldots, 2n\} \setminus \{j+n\}$ is contained in a facet, we have

$$\hat{\chi}_1^*(\mu, n+1, \ldots, 2n) = \hat{\chi}_2^*(\mu, n+1, \ldots, 2n).$$

This gives

$$\chi_1^* = \hat{\chi}_1^* / \{n+1, \ldots, 2n\} = \hat{\chi}_2^* / \{n+1, \ldots, 2n\} = \chi_2^*$$

which proves the theorem. □

This theorem shows that for every non-realizable d-chirotope χ with n vertices there exist two non-simplicial non-polytopal matroid spheres $\mathrm{FL}(\hat{\chi}^*)$ and $\mathrm{FL}((\hat{\chi}^*)^*)$ of dimension $2n-d-2$ and $n+d-2$, respectively, with $2n$ vertices.

For the smallest non-realizable chirotope of rank 4, the Vamos matroid [5] (with 8 vertices), we obtain a non-polytopal 10-sphere with 16 vertices.

Now we formulate the main result of this section:

6.2. THEOREM. *There is a non-polytopal simplicial matroid k-sphere with n vertices for every $k \geqq 3$ and $n \geqq k+6$.*

PROOF. We show that the 3-sphere M^9_{963}, see [2], with 9 vertices has these properties. Then the theorem follows by the matroidal versions of subdividing facets and constructing pyramids, compare [20].

The facets of M^9_{963} are as follows:

1235	$\underline{1289}$	$\underline{2345}$	2689	$\underline{3689}$
$\underline{1237}$	$\underline{1356}$	$\underline{2347}$	$\underline{3459}$	$\underline{4567}$
1256	$\underline{1368}$	2456	$\underline{3478}$	4579
1268	1378	2467	3489	4789
1279	1789	2679	$\underline{3569}$	5679

With a computer program it was shown in [3] that there is (up to a factor ± 1) exactly one 5-chirotope χ^9_{963} with 9 vertices such that $\mathrm{FL}(\chi^9_{963}) = M^9_{963}$.

χ^9_{963} is given by the following list of orientations $\chi^9_{963}(\Lambda(9,5))$ (linewise lexicographically):

$$\begin{array}{l}
-1\ -1\ +1\ -1\ -1\ +1\ +1\ +1\ +1\ +1\ +1\ +1\ -1\ -1\ +1\ +1\ +1\ +1\ +1\ +1\ +1 \\
+1\ -1\ -1\ +1\ +1\ +1\ +1\ -1\ -1\ +1\ -1\ -1\ +1\ +1\ -1\ -1\ -1\ -1\ -1\ -1\ -1 \\
+1\ +1\ +1\ -1\ -1\ -1\ +1\ +1\ -1\ +1\ +1\ -1\ +1\ +1\ +1\ +1\ +1\ +1\ -1\ +1\ +1 \\
-1\ +1\ +1\ +1\ -1\ +1\ +1\ -1\ -1\ -1\ -1\ -1\ -1\ -1\ +1\ +1\ +1\ -1\ -1\ -1\ +1 \\
+1\ -1\ +1\ +1\ -1\ +1\ +1\ +1\ +1\ +1\ +1\ -1\ +1\ +1\ -1\ +1\ +1\ +1\ -1\ +1\ +1 \\
+1\ +1\ +1\ +1\ +1\ +1\ +1\ +1\ +1\ +1\ +1\ +1\ +1\ -1\ -1\ +1\ +1\ -1\ -1\ -1\ -1
\end{array}$$

Using the following Grassmann–Plücker relations multiplied by certain determinants

$$\begin{array}{l}
-\ (12379)\,(36789)\,(12369)\,\{367 \mid 2459\} \\
+\ (12379)\,(36789)\,(34679)\,\{236 \mid 1579\} \\
-\ (12379)\,(12367)\,(34679)\,\{369 \mid 2578\} \\
+\ (12367)\,(35679)\,(34679)\,\{239 \mid 1678\} \\
-\ (12367)\,(35679)\,(12369)\,\{379 \mid 2468\} \\
+\ (35679)\,(36789)\,(12369)\,\{237 \mid 1469\} \ = \ 0
\end{array}$$

One again obtains zero for the sum. On the other hand, pairwise, some of the summands can be cancelled to give the identity

$$\begin{array}{l}
-\ (12379)\,(36789)\,(12369)\,(34567) \\
-\ (12379)\,(36789)\,(34679)\,(12356) \\
+\ (12379)\,(12367)\,(34679)\,(35689) \\
-\ (12367)\,(35679)\,(34679)\,(12389) \\
-\ (12367)\,(35679)\,(12369)\,(34789) \\
+\ (35679)\,(36789)\,(12369)\,(12347) \ = \ 0.
\end{array}$$

Considering the orientations of the chirotope all these products (leading sign included) must have equal sign contradicts this identity. Since all determinants contain point 3 we see that already the 4-chirotope $\chi^9_{963}\{3\}$ with 8 vertices cannot be realizable.

It is known that for every rank $d \geqq 3$ there is an infinite family of minor-minimal non-realizable chirotopes [9]. Hence an easy combinatorial realizability criterion by excluding a finite number of subchirotopes cannot exist in the context of oriented matroids.

It seems likely that a similar result holds for the polytopality of spheres. This would imply that a local Steinitz theorem does not exist for higher dimensions.

6.3. Problem. For fixed $k \geqq 3$ does there exist an infinite set $\mathcal{S}$ of spheres, such that for every $S \in \mathcal{S}$:

(1) S is non-polytopal;

(2) for every vertex x of S:

(2.1) the vertex figure S/x is polytopal,

(2.2) the antistar $\mathrm{ast}(x, S)$ of x can be embedded in a polytopal sphere.

It seems impossible to construct such a family from the known examples of non-realizable oriented matroids, because all Lawrence-like constructions yield increasing rank.

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